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Multiple-trace operators and non-local string theories

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Abstract: We propose that a novel deformation of string perturbation theory, involving non-local interactions between strings, is required to describe the gravity duals of field theories deformed by multiple-trace operators. The new perturbative expansion involves a new parameter, which is neither the string coupling nor the coefficient of a vertex operator on the worldsheet. We explore some of the properties of this deformation, focusing on a special case where the deformation in the field theory is exactly marginal.

Keywords: Superstrings and Heterotic Strings, Superstring Vacua, Conformal Field Models in String Theory, AdS-CFT Correspondance.

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Contents

1. Introduction

Perturbative string theory provides us with a rigid set of rules for computing S-matrix elements in particular limits of M Theory. The building blocks of these rules are the genus expansion of Riemann surfaces, and a conformal field theory on each of these surfaces. Although this is a very rich structure, which is only partially understood, one wonders whether this is the most general set of rules, or just the tip of the iceberg. There are, of course, many backgrounds for which we have no systematic perturbative description; this is the case for generic backgrounds in M theory. There are also decoupling limits of M theory which give string theories which are inherently strongly coupled, such as "little string theories". However, all previously studied backgrounds which are amenable to a perturbative description can be described by the usual set of rules.

In this paper we will discuss new backgrounds of critical string theory, which have a good perturbative description but which require an enlarged set of rules. To construct these backgrounds we will use a specific kinematical setting, that of the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence, which has already taught us many surprising facts about string theory. This will allow us to give a non-perturbative definition (and, in particular, a strong argument for existence) of these backgrounds. The usual set of rules involves a fixed local conformal theory on the worldsheet and a perturbative expansion in powers of the string coupling. In the enlarged rules we will have an additional parameter, which does not correspond to the string coupling or to a local vertex operator on the worldsheet. One can think of the new parameter as the weight of new forms of degenerate worldsheets, which roughly correspond to zero size worm-holes in the 2D gravity of the worldsheet; this is analogous to the way in which the string coupling is the weight of a handle on the worldsheet. Alternatively, we can try to sum the perturbation theory in the new parameter, and then we remain with the usual genus expansion but with a non-local worldsheet action, including interactions between disconnected components of the worldsheet.

The kinematical context that we will be discussing is a deformation of string theory on an AdS_5 space, which is holographically dual to a four dimensional conformal field theory [1]–[4]. Deformations of the conformal field theory by single-trace operators have been extensively studied in the last few years, and we wish to generalize the discussion to deformations of the conformal field theory by multiple-trace operators. From the field theory point of view such a deformation is not significantly different (at finite N) from a single-trace deformation, but we will see that from the point of view of string theory they seem quite different. Our main example will be based on type-IIB string theory on $AdS_5 \times T^{1,1}$, which was discussed in [5]–[8]. The corresponding $\mathcal{N} = 1$ CFT is relatively well understood, and one can show that it contains an exactly marginal deformation which is a superpotential term involving a product of two gauge-invariant chiral operators (whose dimensions add up to 3). Phrased in the language of an asymptotically free UV theory which flows to this CFT, the operator we deform by can be written as a product of two traces. Double-trace perturbations can also be generated radiatively as in the examples of [9, 10], where there is again a family of possible coefficients determined in these cases by a choice of renormalization-group trajectory in the field theory.

In string theory on AdS space this presents the following puzzle: on the one hand one expects to be able to deform the string theory background to include the double-trace deformation, but on the other hand, there is no obvious parameter in conventional string theory corresponding to such a deformation. The usual parameters of string theory involve turning on vertex operators on the worldsheet, which in AdS is the same as changing the VEV of a field in spacetime; but this corresponds to the deformation of the field theory by a single-trace operator (or a simple generator of the chiral ring). Note that, as frequently done in discussions of string theory in RR backgrounds, we are assuming that the relevant conformal field theories behave in a standard way, as described above, though they are not well-understood.

Therefore, it is clear that these examples with double-trace operators (or, more generally, multiple-trace operators) appearing in the lagrangian lead to a prediction for a novel form of perturbative string theory on the gravity side. In this paper we will explain some aspects of the new perturbative expansion, and some of its surprising features in the bulk of the target space. We will see that the resulting string theories are non local both on the worldsheet and in space-time, so we dub them "non-local string theories" (NLSTs).

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We should emphasize that the role of *states* involving multiple-trace operator excitations around the ordinary unperturbed AdS/CFT background is well known: they describe multiparticle states on the gravity side (see, e.g. [4]), and the anomalous dimensions of the operators creating these states have been computed in some examples. Our goal here is to articulate properties of the gravity side of the correspondence when we *perturb the lagrangian* by multiple-trace operators.

Double-trace perturbations of matrix models for non-critical strings were studied in e.g. [11]–[17], where it was observed that the presence of such terms seems to lead to contact interactions on the string worldsheet. In the matrix model context it was conjectured that these terms may be interpreted as changing the branch of the Liouville dressing; the relation between this case and the critical string case we discuss here is not clear. The question of what a multi-trace deformation would mean on the gravity side of AdS/CFT dual pairs was raised previously in [18].

The organization of the paper is the following. We begin in section 2 by discussing general properties of field theories with double-trace (and multiple-trace) deformations in the 't Hooft large-N expansion. In section 3 we discuss the details of some specific examples where multiple-trace perturbations arise on the field theory side of an AdS/CFT dual pair. Using the field theory results, we proceed in section 4 to give a worldsheet description of the new "genus-wormhole" expansion in a perturbative expansion around the undeformed system, and discuss some of the features of the deformation from the point of view of the gravitational dual theory, such as its non-locality. We also discuss briefly how one might approach a more generic description of this type of background. In section 5 we discuss an alternative approach to the problem, in which we view the deformation as changing aspects of boundary conditions on AdS.

Although we will exhibit the non-local deformation only for certain AdS/CFT dual pairs, it raises the interesting question of whether this sort of string perturbation theory might exist in (and define) more general backgrounds. Such more general backgrounds could lead to new low-energy effective actions. It is clear that the deformation does not depend on the presence of RR background fields, since it exists also in AdS_3 cases with NS-NS backgrounds, which we hope to study in future work. However, it is not clear what are the requirements on the asymptotic geometry for this type of deformation to make sense. It should also be interesting to study the effects of the non-local behavior we identify here on the calculation of UV-sensitive quantities on the gravity side such as the vacuum energy, and on non-local operators such as Wilson lines on the field theory side. Because of the novelty of this sort of system, our analysis here is rather preliminary, but we hope that our observations will help to stimulate further development of these theories, and, perhaps, the construction of a more general picture of what types of perturbative string theories exist (and appear in some corners of the M theory moduli space).

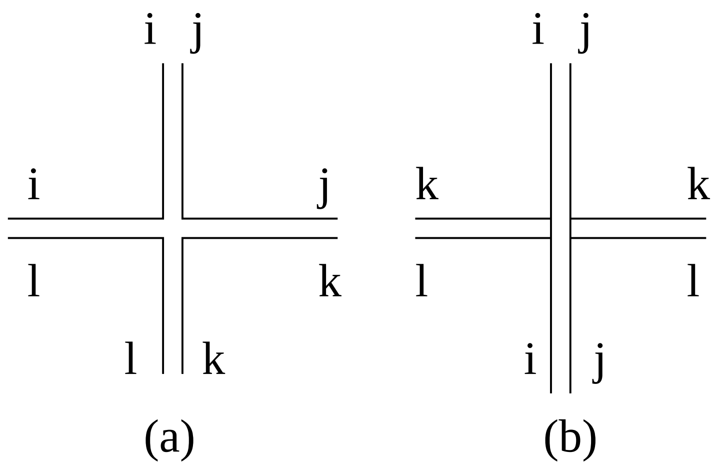

Figure 1: Examples of vertices (in double line notation) corresponding to (a) single-trace and (b) double-trace operators, where the indices run over $1, \ldots, N$.

2. The large-N limit of double-trace operators in field theory

SU(N) gauge theories in which all fields are in the adjoint representation (or in bifundamental representations) usually have a lagrangian involving only terms which can be written as a single trace, like the standard gauge kinetic term $\mathrm{tr}(F_{\mu\nu}^2)$. If one normalizes the fields such that the lagrangian is proportional to $1/g_{\mathrm{YM}}^2$, there exists a 't Hooft large-N limit in which $\lambda \equiv g_{\mathrm{YM}}^2 N$ is kept constant. In this limit the perturbation theory becomes a double expansion in $1/N^2$ and in λ (see, e.g. [19]). Diagrams which have genus g (in the standard double line notation for the Feynman diagrams) contribute with a factor of N^{2-2g} times some power of λ. In particular, in a standard normalization for the single-trace operators $\mathcal{O}_i$ in the theory, in which $\mathcal{O}_i$ is N times a trace of a product of the fields (up to some function of λ), the correlation functions of $\mathcal{O}_i$ have an expansion of the form

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_j \rangle = \sum_{g=0}^{\infty} N^{2-2g} f_g(\lambda) \,. \tag{2.1}$$

The first term in the sum, which corresponds to the planar diagrams, dominates in the large-N limit. In (2.1) we only wrote down the contribution from connected diagrams, of course there are also disconnected diagrams whose generating function is the exponential of the generating function for the connected diagrams.

The building blocks of the diagrammatic expansion described above are the single-trace vertices, which can be written in the plane as described in figure 1*(a)*. On the other hand, if we have a vertex corresponding to a double-trace operator (such as $\mathrm{tr}(F_{\mu\nu}^2)^2$), or an insertion of such an operator, then this cannot be drawn in the plane, as in figure 1*(b)*. Thus, naively it seems that the contribution of double-trace operators will always be negligible in the large-N limit, since they should only appear in non-planar diagrams.

However, there is also the possibility that the double-trace operator can connect two parts of the Feynman diagram which were originally disconnected, as in figure 2. If instead of the double-trace operator we had two single-trace operators (one in each connected component), whose coupling constant is taken to be proportional to N in the 't Hooft large-N limit, then this diagram would scale as $(N^2)^2$ (a product of two standard planar contributions). Therefore, when we have the double-trace vertex instead, such a diagram will scale as N^2 times the double-trace coupling constant.

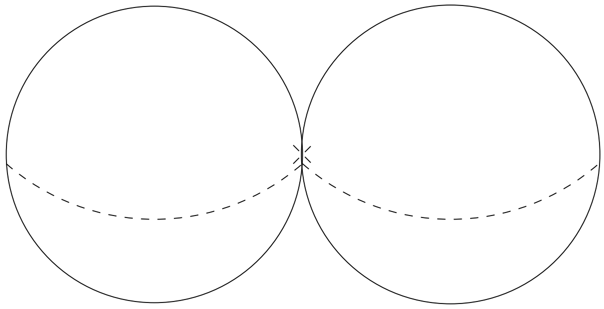

Figure 2: A double-trace vertex connecting two components of a Feynman diagram that was originally disconnected.

If we scale the double-trace coupling constant such that it grows with N in the large-N limit, then this type of diagram, which becomes disconnected when one removes the double-trace operators, will dominate in the large-N limit, since its contribution would grow faster than N^2. Such a large-N limit would not correspond to a string theory, but perhaps to some branched polymer theory, and we will not discuss it here. Instead, we will take the double-trace coupling constant to be some constant $\tilde{h}$ in the large-N limit.[1] Then, it is easy to see that the leading large-N limit will be given by the sum of diagrams with $k+1$ spheres which are connected only by k double-trace vertex operators, giving a contribution which scales as $\tilde{h}^k N^2$ (see figure 3). The subleading contributions in N will involve both non-planar diagrams and diagrams which are connected to themselves by double-trace operators. The contribution of any diagram may be simply computed by counting the number k of double-trace operators (corresponding to a $\tilde{h}^k$ contribution) and then replacing the double-trace operator by a small throat joining smoothly the two worldsheets which it connects. A diagram which after this replacement has genus g will scale as N^{2-2g} in the large-N limit.[2]

The full expression for the correlation functions in a theory which has such a double-trace coupling will be of the form

$$\langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_j \rangle = \sum_{g=0}^{\infty} N^{2-2g} \sum_{k=0}^{\infty} \tilde{h}^k f_{g,k}(\lambda) \,. \tag{2.2}$$

[1]If we take the coupling constant to decrease as a power of N, the diagrams involving the double-trace operators are negligible in the large-N limit, so we get the same large-N string theory in the zero string coupling limit.

[2]It is easy to generalize this analysis also to vertices involving a product of n traces for higher n; the coefficients of such vertices need to scale as N^{2-n} in the large-N limit to have a good (string-like) large-N expansion in which these vertices affect the leading order terms in the genus expansion.

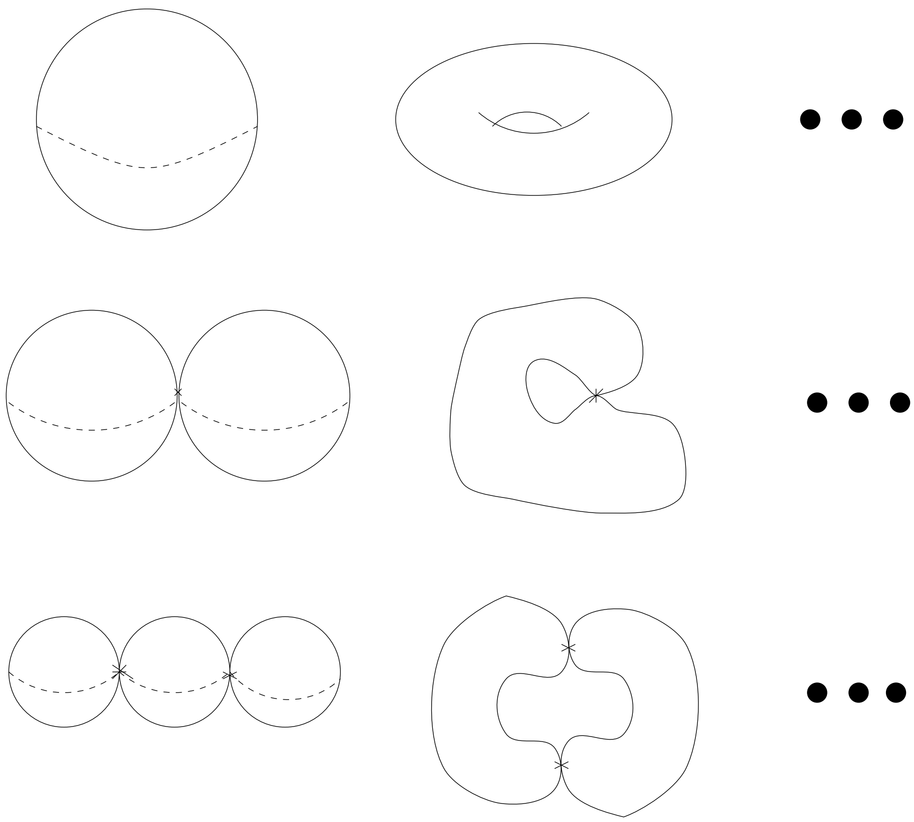

Figure 3: Various diagrams with standard vertices and double-trace vertices. If $\tilde{h}$ denotes the double-trace coupling, then the diagram in the i'th row ($i = 0, 1, \ldots$) and the j'th column ($j = 0, 1, \ldots$) will scale as $\tilde{h}^i N^{2-2j}$ in the large-N limit we are considering.

Note that since we are adding a local double-trace operator, the two worldsheets which are being connected by the vertex actually touch at the position of the double-trace vertex in the four dimensions along which the QFT lives.[3] Thus, it seems that the new interactions (governed by the perturbation theory in $\tilde{h}$)should be realized as contact interactions between worldsheets from the point of view of the string theory describing the large-N limit of the field theory, as in figure 2. As we will see below, in our AdS/CFT cases where the dual string theory lives in more dimensions, this picture will be modified.

Since the leading diagrams in the large-N limit are almost disconnected, it is easy to compute them in terms of correlation functions in the original theory before we added the double-trace operator. For example, suppose we add to a lagrangian involving only single-trace operators a term of the form

$$\delta\mathcal{L} = \int d^4x \frac{\tilde{h}}{N^2} \mathcal{O}_{n_1}(x) \mathcal{O}_{n_2}(x) \,, \tag{2.3}$$

[3]We will limit ourselves here to four dimensional field theories, though everything we say applies to any dimension.

for some single-trace operators $\mathcal{O}_{n_1}$ and $\mathcal{O}_{n_2}$. Then, the large-N limit of any correlation function will be of the form

$$\langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_j\rangle = \langle \mathcal{O}_1\mathcal{O}_2\cdots\mathcal{O}_j\rangle_0 + \frac{\tilde{h}}{N^2}\sum_{\text{partitions}}\int d^4x \langle \mathcal{O}_{i_1}\mathcal{O}_{i_2}\cdots\mathcal{O}_{i_k}\mathcal{O}_{n_1}(x)\rangle_0 \times \\ \times \langle \mathcal{O}_{n_2}(x)\mathcal{O}_{i_{k+1}}\mathcal{O}_{i_{k+2}}\cdots\mathcal{O}_{i_j}\rangle_0 + \\ + O\left(\tilde{h}^2\right), \tag{2.4}$$

where $\{i_1, i_2, \ldots, i_k; i_{k+1}, \ldots, i_j\}$ is some partition of the numbers from 1 to j into two groups, and $\langle\rangle_0$ are the correlation functions in the theory before the deformation. Note that all terms in (2.4) scale as N^2 in the large-N limit; there are also terms of higher order in $1/N^2$ which we did not write down explicitly.

3. Explicit realizations of double-trace perturbations

(a) Running $\tilde{h}$ Usually we do not discuss double-trace operators in lagrangians since (at least in four dimensions) they tend to be irrelevant, so we cannot just add them to the lagrangian. However, such terms can, and generally do, arise when we integrate out other fields. For example, in the theories studied in [9, 10], double-trace quartic scalar perturbations are generated radiatively in QFTs arising at low energies on D-branes at codimension < 6 orbifold fixed planes. In this case the corresponding single-trace perturbations are absent at large N due to symmetries and inheritance of $\mathcal{N} = 4$ supersymmetric non-renormalization theorems. When double-trace operators arise from integrating out fields, they necessarily arise with physical coefficients — including bare contributions plus possible counterterms — scaling as described in the previous section (or smaller in the large-N limit, depending on the genus of the diagram at which they first appear). In these theories, there is a choice of QFT determined by the specific renormalization condition chosen, which determines the physical double-trace coupling at a particular subtraction point M. Different choices are related by finite shifts in counterterms, which amount to finite shifts in $\tilde{h}$ at scale M.

This case has the nice feature that there is a weakly-coupled limit of the theory in which the double-trace interaction is evident perturbatively [9, 10]. However, it has the complications of broken supersymmetry and running couplings, and it is not clear precisely what is the string theory dual of this case and how its parameters are related to the field theory parameters. Therefore, we will largely focus on a strongly coupled but supersymmetric system in which there is an exactly marginal double-trace perturbation.

(b) The exactly marginal case We will focus here on the following simpler case, where one can show that there exists an exactly marginal double-trace operator. An advantage of this case, beyond the fact that the operator is exactly marginal, is that the single-trace operators involved are both chiral, so there is no singularity when we bring them together and no subtlety in defining the double-trace operator (which is just $\lim_{x\to y}\mathcal{O}_{n_1}(x)\mathcal{O}_{n_2}(y)$). The example we will discuss is a deformation of the one discussed by Klebanov and Witten in [5]. Let us start from an $\mathrm{SU}(N)\times\mathrm{SU}(N)$ $\mathcal{N}=1$ supersymmetric gauge theory, with two bifundamental chiral superfields A_i ($i=1,2$) and two anti-bifundamental chiral superfields B_j ($j=1,2$). This theory is believed to flow in the IR to a strongly coupled fixed point; in fact, using the NSVZ formula for the beta functions, one can show that for any value of the couplings the beta functions for the two gauge couplings g_1 and g_2 are the same, so one expects to have a one-dimensional surface of fixed points, which are solutions to the equation $\beta(g_1,g_2)=0$. If we now deform the theory by adding a superpotential

$$W = Nh\,\mathrm{tr}\,(A_1B_1A_2B_2 - A_1B_2A_2B_1) \tag{3.1}$$

(which preserves the $\mathrm{SU}(2)\times\mathrm{SU}(2)\times U(1)_R$ global symmetry of the original superconformal field theory), then one can show [5] by the methods of [20] that the beta function for h is proportional to that of g_1 and g_2, so that there is still just one equation $\beta(g_1,g_2,h)=0$ which needs to be satisfied for the theory to be conformally invariant. Thus, we expect to have a two dimensional manifold of exactly marginal deformations for these conformal theories, and this was verified also in the AdS dual of these theories in [5].

Now, we can ask what happens if we deform the theory described above by an additional superpotential of the form

$$\delta W = \tilde{h}\,(\mathrm{tr}\,(A_1B_1)\,\mathrm{tr}\,(A_2B_2) - \mathrm{tr}\,(A_1B_2)\,\mathrm{tr}\,(A_2B_1))\,, \tag{3.2}$$

which also preserves the same global symmetries. It is easy to see that the anomalous dimension of this operator is the same as that of the operator in (3.1), so the beta functions of h and $\tilde{h}$ are proportional to each other (as well as to the beta functions of the gauge couplings). Therefore, we expect to still have a solution to the equation $\beta=0$ even when we turn on the coupling $\tilde{h}$, so this is an exactly marginal coupling. In the scaling we wrote here the large-N limit behaves as described in the previous section. For small $\tilde{h}$ we will thus have in the large-N limit a double perturbation expansion in $1/N^2$ and in $\tilde{h}$, as described above.[4]

Note that if we write the lagrangian corresponding to (3.2) in components, the linear term in $\tilde{h}$ is indeed a sum of various double-trace operators (including terms where each trace is either bosonic of fermionic), but there is also the scalar potential (after we integrate out the auxiliary fields) which includes a triple-trace operator, whose coefficient is proportional to $|\tilde{h}|^2/N$ (where the $1/N$ comes from the Kähler metric in our normalization). Such a term behaves in the large-N limit similarly to

[4] We could also have a perturbation expansion in h, but in view of the AdS/CFT application we have in mind we prefer not to take h to be small, but just to be constant in the large-N limit. The supergravity approximation in this case is only valid at large h.

the double-trace term we described in detail above (as would a term with a product of k traces and a coefficient $\tilde{h}^{k-1}/N^{k-2}$). We cannot write down this term explicitly because we do not know the form of the Kähler metric in this strongly coupled theory, but eq. (3.2) uniquely specifies the deformation we are performing.

If we wish, we can alternatively describe this deformation by using an auxiliary singlet field λ_{ij} in the $(\mathbf{2},\mathbf{2})$ representation of the global $\mathrm{SU}(2) \times \mathrm{SU}(2)$ symmetry group. Then, we can write instead

$$\delta W = \epsilon^{ik}\epsilon^{jl}\left(\lambda_{ij}\,\mathrm{tr}\,(A_k B_l) - \frac{1}{2\tilde{h}}\lambda_{ij}\lambda_{kl}\right). \tag{3.3}$$

Integrating out the auxiliary field λ_{ij} reproduces the action (3.2). One can think of the contact interaction between worldsheets described in the previous section as arising from integrating out the infinitely massive auxiliary field λ_{ij}.

For small $\tilde{h}$ the correlation functions of the theory will change by a small amount, given by (2.4) (including contributions from all the double-trace operators in our lagrangian, and at higher orders also from the triple-trace operator). However, the moduli space of the theory changes in a more drastic way, since some of it is lifted by the new term (3.2). The moduli space of the original theory is [5] a symmetric product of N copies of the conifold $z_{11}z_{22} - z_{12}z_{21} = 0$. On the moduli space we can diagonalize all the matrices A_1, A_2, B_1 and B_2 (in some arbitrary basis), and the gauge-invariant combinations (up to permutations) are $z_{ij}^{(I)} \equiv (A_i)_{II}(B_j)_{II}$ which can be identified with positions on the conifold.

After we deform the theory, pairs of eigenvalues interact strongly at long distance. If only the first eigenvalue of the matrices is non zero, the potential still vanishes. However, if two eigenvalues are non zero, $\partial W/\partial A_1$ includes terms proportional to $\tilde{h}(A_2)_{II}((B_1)_{22}(B_2)_{11} - (B_1)_{11}(B_2)_{22})$, which lead to pairwise forces which grow with distance on the conifold. One way to cancel all these forces is to impose the four complex conditions $\mathrm{tr}(A_iB_j) \equiv 0$. This suffices to cancel the corrected F-terms, giving a branch of the moduli space of real dimension $6N - 8$.[5] There is another branch of the moduli space, of dimension $6 + 2(N-1)$, in which this constraint does not necessarily hold, but the k'th eigenvalue of all the matrices is given by α_k times the first eigenvalue for some complex number α_k.

4. Double-trace operators in the AdS/CFT correspondence

4.1 General properties

The theories discussed in the previous section, before including the double-trace deformation, have dual descriptions, using the AdS/CFT correspondence [1]–[4], as type-IIB string theory on $AdS_5 \times K$ for Einstein spaces K, where $K = S^5/Z_n$ in the examples of section 3a, and $K = T^{1,1}$ in the example of section 3b [5] (where

[5] We thank D. Berenstein for pointing out this branch of the moduli space.

$T^{1,1} = (\mathrm{SU}(2) \times \mathrm{SU}(2)/U(1))$ is the base of the cone in the conifold geometry). Thus, we can ask what these deformations correspond to in the dual theory. Recall that in general, every single-trace operator in the field theory corresponds to a field propagating on AdS space, and to a vertex operator in the corresponding worldsheet string theory. Deforming the field theory by a single-trace operator is dual to turning on the corresponding vertex operator in the string theory, or (equivalently) to looking at configurations in which the corresponding field on AdS has a particular behavior near the boundary. For example, deforming the field theory of section 3b by the two exactly marginal single-trace operators corresponds in the string theory dual to changing the (constant over all space) value of the string coupling and axion and of the integral of the 2-form fields over the non-trivial 2-cycle in $T^{1,1}$.

On the other hand, double-trace operators do not correspond to fields on AdS or to vertex operators in the string theory. In some sense they correspond to two-particle configurations on AdS, since we can define the double-trace operator by its appearance in the OPE of two single-trace operators. This suggests that perhaps deforming by such an operator should be described by some two-particle condensate in the bulk (a "squeezed state") behaving in a particular way near the boundary, analogous to the description of single-trace deformations. One can perhaps construct such a state as a coherent state in an off-shell formulation of the low-energy supergravity, but it is not clear how to promote this description to the full string theory, which does not contain any off-shell information in the bulk. Thus, we will propose an alternative description for this deformation based on (a generalization of) worldsheet string theory.

An important property of the double-trace deformation we are discussing is that, unlike the single-trace deformations, the double-trace deformation leads to a theory which is non local in the ten dimensional bulk:

1. One way to see the non-locality is from the description above as a coherent state of two particles, each of which is in some spherical harmonic on K; obviously this induces long-term correlations between different positions on K.

2. The non-locality on K also follows from the form of the corrections (2.4) to correlation functions. Correlation functions of particular spherical harmonics will change in a different way from those of other spherical harmonics; for example, only the two point functions of the low-lying spherical harmonics corresponding to the operators explicitly appearing in the double-trace term will change. This again suggests that the deformation has no local description in ten dimensions.

3. Some of the original moduli space before we turned on the deformation corresponded to configurations of D3-branes sitting at arbitrary radial positions and arbitrary positions in K. After the deformation, as described above, we

can still have a single D3-brane and no force will act on it, but if we have two D3-branes at different positions on K there should be a force between them that grows with the distance (along K), coming from the new scalar potential after the deformation. Again, this is inconsistent with a local description in ten dimensions. Formulas for similar long-range potentials can be found in [10] for case (a).

Note that the scale of non-locality suggested by these observations is the radius R of K, which is also the radius of curvature of the AdS space; we will mostly be working in the limit where this is much larger than the string scale (at which the theory is obviously non local). This requires us to consider case (b) and take the coupling h discussed above to be large.

The arguments above suggest that after the deformation the theory is non local on K, but it could still be local on the AdS space. The fact that the deformation is local in $\mathbb{R}^4$ suggests that it might be local also in AdS, but this is also consistent with a non-locality of the order of the AdS radius (since it is hard to describe smaller objects than this in the field theory). In our description below we will see that the resulting theory seems to be non local also on AdS.

4.2 The deformation in string theory

How can we describe the deformed theory? Usually the only deformations we are allowed to do in string theory have a perturbative description involving turning on vertex operators on the worldsheet. In conformal perturbation theory, one adds

$$\delta S = \epsilon \int d^2\sigma \sqrt{\hat{g}} V \tag{4.1}$$

to the worldsheet action, which preserves conformal invariance if V is an exactly marginal physical vertex operator. Working in the original theory and adding contributions obtained by bringing down powers of δS into correlation functions produces the corrected correlation functions of the deformed theory, perturbatively in ϵ. For example, if we consider a circle parameterized by target space coordinate X, adding $\int d^2\sigma \sqrt{\hat{g}} \epsilon V \equiv \int d^2\sigma \sqrt{\hat{g}} \epsilon \partial X \bar{\partial} X$ changes the radius squared of the circle by ϵ.

More generally, one can go beyond this conformal perturbation theory by resolving for the spectrum and interactions of the deformed theory using the new worldsheet action $S + \delta S$ (or equivalently by performing the Polyakov path integral with the shifted action). More generally still one can consider terms of the form $\int d^2\sigma \sqrt{\hat{g}} G_{\mu\nu}(X) \partial X^\mu \bar{\partial} X^\nu$ for which the condition for conformal invariance translates into the condition $R_{\mu\nu} = 0$ (at lowest order in α').

In the context of the AdS/CFT correspondence, these standard deformations on the gravity side correspond only to deformations by single-trace operators on the boundary, so the double-trace deformation we are interested in here cannot be described in this way. It seems that it is not possible to describe this deformation from the point of view of a single worldsheet, as suggested also by the field theory analysis above where the deformation corresponds to a contact interaction between worldsheets. We do not know how to describe the deformation at a fundamental level, but we can reproduce the perturbation expansion in $\tilde{h}$ (2.4), analogous to the perturbation theory about a fixed background worldsheet CFT deformed by (4.1). This is accomplished by adding an interaction term to the usual sum over all worldsheets (connected and disconnected), as follows.

Suppose that our interaction in the gauge theory is of the form (2.3), with $\mathcal{O}_{n_1}(x_1)$ of dimension Δ_1 corresponding to a vertex operator of the form $V_{n_1}(\theta(w))f_{\Delta_1,x_1}(x(w), z(w))$, where V_{n_1} includes the appropriate spherical harmonic as a function of the compact space, w is the complex coordinate on the worldsheet,[6] x and z are coordinates on AdS with the metric $ds^2_{\mathrm{AdS}} = \frac{R^2}{z^2}(dz^2 + dx^\mu dx_\mu)$, and f_{Δ_1,x_1} is the non-normalizable wave-function on AdS for an operator of dimension Δ_1 with delta-function support at a point x_1 on the boundary. There is a similar vertex operator corresponding to $\mathcal{O}_{n_2}(x_2)$. We are being schematic here since in any case we do not have a good description of string theory in this RR background. Then, it seems that we need to add an interaction between two worldsheets of the form

$$\begin{aligned}
\delta\tilde{S} &= \tilde{h}\int d^4x \int d^2w_1 V_{n_1}(\theta_1(w_1)) f_{\Delta_1,x}(x_1(w_1), z_1(w_1)) \times \\
&\quad \times \int d^2w_2 V_{n_2}(\theta_2(w_2)) f_{\Delta_2,x}(x_2(w_2), z_2(w_2)) \\
&\equiv \tilde{h}\int d^2w_1 \int d^2w_2 V_{n_1}(\theta_1(w_1)) V_{n_2}(\theta_2(w_2)) K \times \\
&\quad \times (x_1(w_1), z_1(w_1); x_2(w_2), z_2(w_2))\,, \qquad (4.2)
\end{aligned}$$

where

$$K(x_1, z_1; x_2, z_2) = \int d^4x G_{\Delta_1}(x; x_1, z_1) G_{\Delta_2}(x; x_2, z_2)\,, \qquad (4.3)$$

and G_Δ is the boundary-to-bulk propagator on AdS_5, given in euclidean space by

$$G_\Delta(x; x_1, z_1) = \pi^{-2}\frac{\Gamma(\Delta)}{\Gamma(\Delta-2)}\frac{{z_1}^\Delta}{(z_1^2 + (x-x_1)^2)^\Delta}\,. \qquad (4.4)$$

As in our field theory discussion, w_1 and w_2 can either be on the same connected component of the worldsheet or on different connected components. For two points on the boundary, $z_1 = z_2 = 0$, K is just a delta function (as expected from the field theory analysis of section 2, which suggested a contact interaction in 4d), but in the bulk it is non zero also when $x_1 \neq x_2$. Our vertex operators appearing in (4.2) contain an implicit factor of the string coupling g_s, as is standard for vertex operators describing ordinary string excitations (though usually one does not include this factor in the standard deformations (4.1) describing condensation of strings).

[6] When we write $x(w)$ we refer to a general function/operator depending on w and $\bar{w}$.

This yields the correct g_s-dependence to match the field theory expansion (2.2), (2.3) and (2.4) (with the normalization of field theory operators as discussed in the previous sections). In momentum space we can write K as

$$K\left(x_1, z_1; x_2, z_2\right) = \int d^4k e^{ik\cdot(x_1-x_2)} z_1^2 z_2^2 K_{\Delta_1-2}\left(|k|z_1\right) K_{\Delta_2-2}\left(|k|z_2\right) \tag{4.5}$$

in euclidean space; the lorentzian case is defined as usual by analytic continuation of this expression, which involves the same modified Bessel functions $K_\nu(z)$ for $k^2>0$ and Hankel functions $H_\nu(z)$ for $k^2<0$ [21].

The added interaction (4.2) manifestly reproduces the leading term in (2.4), at least in the supergravity approximation. Since the interaction (4.2) that we added is a sum of products of vertex operators (of dimension $(1,1)$) on each worldsheet, it is clear that it preserves conformal invariance. However, since this interaction involves two worldsheets, it does not seem to be equivalent to any standard string interaction which is local on the worldsheet.

Similarly, we can generalize (4.2) to an interaction between three strings that would correspond to a triple-trace operator, which we need to do if we want to add a double-trace operator in a way which preserves supersymmetry (as described in the previous section). Note that (as argued, for instance, in [22]) when we compute sphere correlation functions on AdS, the volume of $\mathrm{SL}(2,\mathbb{C})$ is absorbed by the integral over the radial position on AdS, so two-point functions do not necessarily vanish on the sphere.

There is one simplification evident from the diagrammatics of our deformation that is worth pointing out. Although this deformation is not a modulus of local supergravity, and has various novel non-local features as discussed in section 4.1, it follows from our perturbative formulation in this section that graviton scattering alone is unaffected by the deformation at order $\mathcal{O}(N^2)$ (tree-level on the gravity side). At this order, all diagrams with only gravitons on external legs involve at least one genus zero component of the worldsheet with insertions of a single field ϕ_{n_1} or ϕ_{n_2} (dual to the factors $\mathcal{O}_{n_{1,2}}$ in the double-trace perturbation), combined with some number of gravitons. At tree level, matter fields do not couple linearly to gravitons, and these diagrams all vanish. In the case of the orbifold models of case (a), in fact the self-interactions of all untwisted modes are unaffected by the deformation at $\mathcal{O}(N^2)$. This follows from the inheritance of untwisted amplitudes in orbifold field and string theories at this order.

4.3 Comments on possible generalizations

Next, we can ask if, as in the case of ordinary single vertex operator deformations whose condensation leads to a change in the space-time background appearing in the worldsheet lagrangian, we can find also in this case an exact description going beyond

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our generalization of worldsheet conformal perturbation theory. Perturbatively, we describe our deformation by a Polyakov path integral of the form

$$\sum_{\text{disconnected}} \int [DY] e^{-S_0 - \delta\tilde{S}}\,, \tag{4.6}$$

where S_0 is the worldsheet action before the deformation, and where we use $Y \equiv (\theta, x, z)$ to denote the full set of space-time coordinates. We would like to know if we can describe the deformation more generally using a Polyakov path integral of the form

$$\sum_{\text{disconnected}} \int [DY] e^{-S}\,, \tag{4.7}$$

where

$$\begin{aligned} S = &\sum_{I=1}^{N_w} \int d^2\sigma^{(I)} \sqrt{\hat{g}(\sigma)}\mathcal{L}_0 + \\ &+ \sum_{I=1}^{N_w} \int d^2\sigma_1^{(I)} \sqrt{\hat{g}(\sigma_1)} \sum_{J=1}^{N_w} \int d^2\sigma_2^{(J)} \sqrt{\hat{g}(\sigma_2)} \hat{K}\left[Y(\sigma_1), Y(\sigma_2)\right] + \\ &+ \text{trilocal and higher contributions} \end{aligned} \tag{4.8}$$

for some function $\hat{K}$, where we wrote the worldsheet coordinate σ as a direct sum of contributions from different connected components of the worldsheet: $\sigma = \sum_{I,\oplus} \sigma^{(I)} = \sigma^{(1)} \oplus \sigma^{(2)} \oplus \cdots \oplus \sigma^{(N_w)}$, and N_w is the number of disconnected components of the worldsheet in a given term of (4.7).[7] We have included in (4.8) the possibility of trilocal and higher multilocal terms on the worldsheet, which may be required to cancel violations of Weyl invariance that arise as operators from different bilocal terms in (4.8) collide on a given component of the worldsheet.[8] If this description is to work, we need to know what the conditions on the couplings in S are in order to have conformal invariance. These should follow from requiring cancellation of the anomaly under Weyl rescalings of the worldsheet metric ($\hat{g}_{\alpha\beta} \to e^{2\eta}\hat{g}_{\alpha\beta}$). In other words, if we calculate the Weyl anomaly for the metric using the action (4.8), what conditions do we get on $\hat{K}[Y(\sigma_1), Y(\sigma_2)]$ and on the higher non-local terms in (4.8)? In the linearized approximation one solution for $\hat{K}$ is (4.2)–(4.5), which appears to reproduce the double-trace deformation in conformal perturbation theory. More generally, we might expect to find a much wider class of solutions, analogous to the solutions of the Ricci-flatness condition in the case of the lagrangian $G_{\mu\nu}(X)\partial X^\mu \bar{\partial} X^\nu$ on a single worldsheet. Most of these solutions will not involve products of propagators for single quanta of the bulk fields as we had above, but will instead involve

[7] Similar non-local terms in the worldsheet action were studied on connected worldsheets in [23] in analyzing the problem of D-brane recoil.

[8] Alternatively, we can think of such terms as arising in a worldsheet renormalization group flow starting from a theory with only bilocal couplings.

much more general field configurations $\hat{K}(Y_1, Y_2)$ solving the conformal invariance conditions. Note that in this formalism we still have the usual genus expansion (including also disconnected worldsheets), but with a non-local worldsheet action which relates (possibly disconnected) components of the worldsheet, and changes the power of g_s associated with certain diagrams.

On D-brane probes, there will be new interactions obtained from worldsheets of the sort discussed above but with boundaries on the D-brane probes. In the worldvolume theory, these will in general appear as new interactions. It would be interesting to obtain the rules for what sorts of couplings can consistently be introduced on D-brane probes; perhaps this can be used to constrain the general possibilities for such interactions on the string theory side.

5. An alternative description of the deformation

An equivalent way to describe the deformed theory is by using our description (3.3) of the deformation in the field theory. If we had only the first term in (3.3), then the auxiliary field λ_{ij} appearing there would be identified in string theory with the boundary value for a non-normalizable mode of the field corresponding to the operator $\mathrm{tr}(AB)$. Thus, we can describe the deformed theory by integrating over all possible boundary conditions for this field, with a weight given by $e^{-\frac{1}{2h}\int \lambda_{ij}^2}$ (the precise description is actually a bit more complicated because (3.3) is a superpotential and not a lagrangian, but this raises no new issues). From the bulk point of view, this description may be less useful than the description above, since it involves integrating over boundary conditions and completely obscures the physics in the bulk. However, if we do a perturbation expansion of this description, we recover the description above, where K essentially arises as the inverse propagator for λ_{ij} in the bulk.

If we discuss a theory on an AdS space with a cutoff (which is a UV cutoff in the field theory), then all the couplings, including λ_{ij}, become dynamical fields. We can then introduce interactions like the second term of (3.3) on the cutoff, which will lead to the theories we described above as we take the cutoff to infinity (the kinetic term of λ_{ij} goes to zero in this limit).

One way to view the new interactions is, therefore, in terms of adding an auxiliary field λ_{ij} on the boundary, such that the new interactions all involve this field. Thus, it might seem that we are really not changing the bulk physics at all. However, from the discussion of the previous sections it is clear that (for instance) from the point of view of the low-energy theory the couplings in the bulk change, just like for single-trace deformations. In particular, the double-trace perturbation we are discussing in the conifold case is an exactly marginal perturbation on the field theory side, which therefore affects physics on all scales, and in the cases of section 3a the dynamically generated double-trace contribution grows in the infrared of the field theory. Therefore, on the gravity side these deformations should have effects in the bulk of the AdS space rather than being concentrated at the boundary. Also, one might think that two worldsheets will only interact if a particle can actually be physically transmitted from one to the other through the boundary, but this is clearly not true; there is an infinite potential barrier preventing most normalizable particle excitations in the interior of the space from propagating to the boundary. The bulk-boundary propagator involves instead non-normalizable excitations whose wavefunctions near the boundary are infinitely rescaled relative to those of normalizable excitations, as discussed for example in [24, 25] in the context of the undeformed theory. Furthermore, from the form of (4.3) it seems clear that the new interactions relate also nearby points in the bulk, which cannot be causally connected through the boundary.[9]

It would be interesting to understand if the choice of boundary conditions and path integral weights that we are making can be related to a choice made in taking the near-horizon limit of the full D-brane systems leading to the AdS/CFT dual pairs we are studying. In taking the near-horizon limit, the non-normalizable modes which live in the asymptotically flat region away from the brane freeze out, serving as sources (couplings) in the field theory and background parameters in the gravity theory. In the standard limit [1], the massive fields among them are simply set to zero. But perhaps the near-horizon limit is more subtle in general; it is tempting to speculate that the λ_{ij} appearing in the above prescription are some of these physical asymptotic closed string states, and that there is perhaps some other way of taking the near-horizon limit in which these states, while still not corresponding to dynamical fields, affect the theory differently from what one would get by simply setting them to zero, because of the residual couplings (3.3).

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[9]This arises at least in part from the fact that the propagators appearing in the lorentzian continuation of (4.5) have exponentially decaying contributions outside the light cone. This is of course also a feature of Feynman propagators in ordinary quantum field theory. Here, as there, to diagnose whether acausal effects really arise one would need to study more refined observables such as expectation values of commutators of spacelike separated fields in the bulk.

References

[1] J. Maldacena, *The large-N limit of superconformal field theories and supergravity*, *Adv. Theor. Math. Phys.* **2** (1998) 231 [hep-th/9711200].

[2] S.S. Gubser, I.R. Klebanov and A.M. Polyakov, *Gauge theory correlators from non-critical string theory*, *Phys. Lett.* **B 428** (1998) 105 [hep-th/9802109].

[3] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253 [hep-th/9802150].

[4] O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, *Large-N field theories, string theory and gravity*, *Phys. Rep.* **323** (2000) 183 [hep-th/9905111].

[5] I.R. Klebanov and E. Witten, *Superconformal field theory on threebranes at a Calabi-Yau singularity*, *Nucl. Phys.* **B 536** (1998) 199 [hep-th/9807080].

[6] S.S. Gubser, *Einstein manifolds and conformal field theories*, *Phys. Rev.* **D 59** (1999) 025006 [hep-th/9807164].

[7] S.S. Gubser and I.R. Klebanov, *Baryons and domain walls in an* $N = 1$ *superconformal gauge theory*, *Phys. Rev.* **D 58** (1998) 125025 [hep-th/9808075].

[8] D.R. Morrison and M.R. Plesser, *Non-spherical horizons, 1*, *Adv. Theor. Math. Phys.* **3** (1999) 1 [hep-th/9810201].

[9] A.A. Tseytlin and K. Zarembo, *Effective potential in non-supersymmetric* $\mathrm{SU}(N) \times \mathrm{SU}(N)$ *gauge theory and interactions of type-0 D3-branes*, *Phys. Lett.* **B 457** (1999) 77 [hep-th/9902095].

[10] A. Adams and E. Silverstein, *Closed string tachyons, AdS/CFT and large-N QCD*, hep-th/0103220.

[11] S.R. Das, A. Dhar, A.M. Sengupta and S.R. Wadia, *New critical behavior in* $d = 0$ *large N matrix models*, *Mod. Phys. Lett.* **A 5** (1990) 1041.

[12] F. Sugino and O. Tsuchiya, *Critical behavior in* $c = 1$ *matrix model with branching interactions*, *Mod. Phys. Lett.* **A 9** (1994) 3149 [hep-th/9403089].

[13] S.S. Gubser and I.R. Klebanov, *A modified* $c = 1$ *matrix model with new critical behavior*, *Phys. Lett.* **B 340** (1994) 35 [hep-th/9407014].

[14] I.R. Klebanov, *Touching random surfaces and Liouville gravity*, *Phys. Rev.* **D 51** (1995) 1836 [hep-th/9407167].

[15] I.R. Klebanov and A. Hashimoto, *Nonperturbative solution of matrix models modified by trace squared terms*, *Nucl. Phys.* **B 434** (1995) 264 [hep-th/9409064].

[16] F. David, *A scenario for the* $c > 1$ *barrier in non-critical bosonic strings*, *Nucl. Phys.* **B 487** (1997) 633 [hep-th/9610037].

[17] O. Andreev, *On touching random surfaces, two-dimensional quantum gravity and non-critical string theory*, *Phys. Rev.* **D 57** (1998) 3725 [hep-th/9710107].

[18] G. Arutyunov, S. Frolov and A. Petkou, *Perturbative and instanton corrections to the ope of CPOS in* $N = 4$ SYM_4, *Nucl. Phys.* **B 602** (2001) 238 [hep-th/0010137].

[19] S. Coleman, $1/N$, in the proceedings of the 1979 Erice School on Subnuclear Physics and in *Aspects of symmetry*, Cambridge University Press.

[20] R.G. Leigh and M.J. Strassler, *Exactly marginal operators and duality in four-dimensional* $N = 1$ *supersymmetric gauge theory*, *Nucl. Phys.* **B 447** (1995) 95 [hep-th/9503121].

[21] V. Balasubramanian, P. Kraus and A.E. Lawrence, *Bulk vs. boundary dynamics in anti-de Sitter spacetime*, *Phys. Rev.* **D 59** (1999) 046003 [hep-th/9805171].

[22] D. Kutasov and D.A. Sahakyan, *Comments on the thermodynamics of little string theory*, *J. High Energy Phys.* **02** (2001) 021 [hep-th/0012258].

[23] W. Fischler, S. Paban and M. Rozali, *Collective coordinates for D-branes*, *Phys. Lett.* **B 381** (1996) 62 [hep-th/9604014].

[24] S.B. Giddings, *The boundary s-matrix and the AdS to CFT dictionary*, *Phys. Rev. Lett.* **83** (1999) 2707 [hep-th/9903048].

[25] J. Polchinski, *S-matrices from AdS spacetime*, hep-th/9901076.